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1980 J. Phys. A: Math. Gen. 13 1991

(http://iopscience.iop.org/0305-4470/13/6/021)

J. Phys. A: Math. Gen. **13** (1980) 1991–2000. Printed in Great Britain

Sl(3, *R*) and the repulsive oscillator

P G L Leach

Department of Applied Mathematics, La Trobe University, Bundoora 3083, Australia

Received 31 May 1979, in final form 24 October 1979

Abstract. The complete symmetry group of a particle moving in one dimension under the influence of a negative quadratic potential ('the repulsive oscillator') is shown to be Sl(3, *R*). The generators of the five-parameter subgroup are obtained from the two linear and three quadratic invariants of the Hamiltonian. The additional generators required for the three-parameter subgroup are obtained from the method of extended Lie groups. It is inferred that an n-dimensional, uncoupled, undamped and unforced linear system has the complete symmetry group Sl($n+2$, *R*).

1. Introduction

Discussion of the symmetry groups of dynamical systems, especially classical, has widened in recent years. Initially, discussions were of purely geometrical symmetries, for example rotational invariance. The concept of dynamical symmetry as contrasted to geometrical symmetry arose from the necessity to explain the existence of degeneracies in spectra which were over and above those expected on purely geometric grounds.

For the non-relativistic Kepler problem the conserved Runge–Lenz vector provided additional generators which showed that SO(4) was the appropriate symmetry group. In the case of the non-relativistic isotropic harmonic oscillator, the conserved Jauch–Hill–Fradkin tensor performed a similar task. In this case the symmetry group was SU(3). Each constant of the motion associated with these symmetry groups has zero Poisson bracket with the (appropriate) Hamiltonian. Subsequent development has been in the construction of non-invariance symmetry groups. These have taken the form of non-invariance super-groups as studied by Mukunda *et al* (1965) and of non-invariance groups for time-dependent systems such as those studied by Günther and Leach (1977) and Leach (1978a).

More recently, attention has been given to the complete symmetry groups of dynamical systems. The basic systems studied have been one-dimensional and linear. Linear systems are important physically and have the advantage of being amenable to mathematical treatment. It would appear that the concentration on one-dimensional systems has been from a desire to highlight the symmetry rather than to engage in a demonstration of algebraic dexterity. However, there is a more serious difficulty in the treatment of multi-dimensional linear systems which is related to the diagonalisation of symmetric matrices by symplectic transformations (cf Williamson 1937).

Anderson and Davison (1974) showed that the one-dimensional, time-independent, harmonic oscillator and the free particle both possessed the complete symmetry group Sl(3, *R*). The result for the oscillator was obtained also by Wulfman and Wybourne (1976) who employed the method of extended Lie groups. In an elegant paper, Lutzky (1978) combined Noether's Theorem with a modification of the extended theory to obtain the same result. Leach (1979) showed that the one-dimensional, time-dependent, harmonic oscillator also has $Sl(3, R)$ as its complete symmetry group. The method adopted was based on a combination of linear canonical transformations of the Hamiltonian and the method of extended Lie groups. The complete symmetry group for an n-dimensional time-dependent harmonic oscillator (uncoupled) was shown to be $Sl(n+2, R)$ by Prince and Eliezer (1980). They followed Lutzky's method. By implication the corresponding time-independent problem also possesses $Sl(n+2, R)$ symmetry.

In this paper, the complete symmetry of a particle moving in one dimension under the influence of a negative, time-independent, quadratic potential is shown to be $Sl(3, R)$. With this result established, it may be inferred that the complete symmetry group of an n-dimensional linear system, without damping, coupling or forcing terms, is $Sl(n+2, R)$. This is the case whether the potential terms are time-independent or time-dependent. (The generators for the one-dimensional, time-dependent negative quadratic potential are listed in the Appendix.) It remains to be seen whether the result extends to the three categories of systems which have been excluded.

The main decision to be made when embarking on the determination of a complete symmetry group is which method is to be adopted. In this paper the method used is that which combines the Hamiltonian invariants and the extended Lie theory. The other two methods are believed to be equally suitable in this instance. However, the present writer is not convinced that this will be the case in all instances, especially when applying the method of extended Lie groups. The main motivation for the choice in this problem is that it lies within the author's programme of demonstrating the essential sameness of all classical quadratic Hamiltonians.

The development of the paper reflects that purpose. The canonical transformation from attractive to repulsive oscillator is derived and from this the two linear and three quadratic invariants obtained. Using standard theory the associated generators may then be written down. The form of the remaining three generators is suggested. They are shown to satisfy the partial differential equations arising from the method of extended Lie groups. The eight generators are shown to have the commutation relations appropriate to the symmetry group $Sl(3, R)$, which establishes the result. A comparison of these generators with those of the time-independent and time-dependent harmonic oscillator suggests the generators for the time-dependent repulsive oscillator and they are listed in the Appendix. On a matter of terminology, the system described here is called the 'repulsive oscillator'. On a point of semantics this is, in a sense, nonsensical, but it is believed to be a suitable description.

2. Canonical transformation from attractive to repulsive oscillator

The repulsive oscillator has Newtonian equation of motion

$$\ddot{q} - q = 0 \tag{2.1}$$

and associated Hamiltonian

$$H = \tfrac{1}{2}(p^2 - q^2) \tag{2.2}$$

in which

$$p = \dot{q}. \tag{2.3}$$

In terms of the two-vector $z^{\mathrm{T}}=(q, p)$, the Hamiltonian is

$$H=\tfrac{1}{2}z^{\mathrm{T}}Az \tag{2.4}$$

where the 2×2 real symmetric matrix A is given by

$$A=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{2.5}$$

The attractive oscillator has Hamiltonian

$$\bar{H}=\tfrac{1}{2}\bar{z}^{\mathrm{T}}I\bar{z} \tag{2.6}$$

where I is the 2×2 identity. A linear canonical transformation form $\bar{H}$ to H is accomplished by

$$z=S\bar{z} \tag{2.7}$$

where the 2×2 real matrix S satisfies the system of equations (Leach 1977)

$$\dot{S}=JAS-SJI, \tag{2.8}$$

J being the 2×2 symplectic matrix. The requirement that the transformation be canonical imposes the constraint that

$$SJS^{\mathrm{T}}=J. \tag{2.9}$$

The system of equations (2.8) may be rewritten as

$$\dot{u}=Mu \tag{2.10}$$

where

$$u^{\mathrm{T}}=(S_{11}, S_{12}, S_{21}, S_{22}) \tag{2.11}$$

$$M=\begin{pmatrix}J & I\\ I & J\end{pmatrix}. \tag{2.12}$$

Setting $t_0=0$, the solution of (2.10) is

$$\begin{aligned}u(t)&=\exp(tM)u(0)\\ &=\left\{\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\cosh t\cos t+\begin{pmatrix}J & 0\\ 0 & J\end{pmatrix}\cosh t\sin t\right.\\ &\qquad\left.+\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\sinh t\cos t+\begin{pmatrix}0 & J\\ J & 0\end{pmatrix}\sinh t\sin t\right\}u(0).\end{aligned} \tag{2.13}$$

Writing

$$u(t)=\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}, \qquad u(0)=\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \tag{2.14}$$

the constraint (2.9) becomes

$$u_1^{\mathrm{T}}Ju_2=1. \tag{2.15}$$

As it is desired to express the invariants of $\bar{H}$ in terms of z, the inverse transformation is required. Since

$$\begin{aligned}S^{-1}&=-JS^{\mathrm{T}}J\\ &=-J[u_1(t), u_2(t)]J,\end{aligned} \tag{2.16}$$

when $u_1(t)$ and $u_2(t)$ are substituted from (2.13), the inverse is

$$S^{-1} = (I \sin t - J \cos t)(-u_1 \sinh t - u_2 \cosh t, u_1 \cosh t + u_2 \sinh t). \tag{2.17}$$

3. The invariants

The Hamiltonian (2.6) has five invariants (see Leach (1979) *The Complete Symmetry Group of a One-dimensional Forced Harmonic Oscillator* (unpublished)), two linear and three quadratic in the canonical variables. (There are, of course, invariants of higher degree, but their Poisson bracket relations, which generate more invariants, do not constitute a closed set and so are not suitable to provide the finite number of generators which is a feature of second-order equations.) The linear invariants are given by the elements of the vector

$$C_1 = (I \cos t - J \sin t)\bar{z}. \tag{3.1}$$

The quadratic invariants are given by the elements of the matrix

$$\begin{aligned} C_2 &= C_1 C_1^{\mathrm{T}} \\ &= (I \cos t - J \sin t)\bar{z}\bar{z}^{\mathrm{T}}(I \cos t + J \sin t). \end{aligned} \tag{3.2}$$

There are three linearly independent elements of C_2. The usual forms of the invariants are given by

$$[C_2]_{12} = [C_2]_{21}, \qquad \tfrac{1}{2}([C_2]_{11} + [C_2]_{22}), \qquad \tfrac{1}{2}([C_2]_{11} - [C_2]_{22}).$$

However, the expression in (3.2) is suitable for the present formalism. At the appropriate stage in the development, the invariants will be regrouped.

Applying the transformation

$$\bar{z} = S^{-1} z \tag{3.3}$$

with S^{-1} as given in (2.7), the linear invariants become

$$C_1 = -J(u_1 \cosh t + u_2 \sinh t, u_1 \sinh t + u_2 \cosh t)Jz. \tag{3.4}$$

This may be written in a form free of the vectors u_1 and u_2 by premultiplying by $S(0)$. Then

$$\begin{aligned} C_1' &= S(0) C_1 \\ &= \begin{pmatrix} u_1^{\mathrm{T}} \\ u_2^{\mathrm{T}} \end{pmatrix} C_1 \\ &= (I \cosh t - K \sinh t) z \end{aligned} \tag{3.5}$$

where the matrix K is

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

Under the transformation the matrix of quadratic invariants becomes

$$\begin{aligned} C_2 = J(u_1 \cosh t + u_2 \sinh t, u_1 \sinh t + u_2 \cosh t)Jzz^{\mathrm{T}}J^{\mathrm{T}} \\ \times (u_1 \cosh t + u_2 \sinh t, u_1 \sinh t + u_2 \cosh t)^{\mathrm{T}} J^{\mathrm{T}}. \end{aligned} \tag{3.7}$$

This may be converted to an invariant matrix free of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ by premultiplying by $\boldsymbol{S}(0)$ and postmultiplying by $\boldsymbol{S}(0)^{\mathrm{T}}$. Thus

$$
\begin{aligned}
C_2' &= \boldsymbol{S}(0) C_2 \boldsymbol{S}(0)^{\mathrm{T}} \\
&= (I \cosh t - K \sinh t) \boldsymbol{z}\boldsymbol{z}^{\mathrm{T}} (\mathrm{I} \cosh t - K \sinh t),
\end{aligned}
\tag{3.8}
$$

a result which could have been anticipated by the expression for $\boldsymbol{C}_1'$ in (3.5). It should be emphasised that the rearrangement of $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ to obtain $\boldsymbol{C}_1'$ and $\boldsymbol{C}_2'$ has nothing to do with the canonical transformation from $\bar{H}$ to H. The point is that, for all possible choices of the parameters of the transformation, the same set of invariants suffices. Hereafter the prime is dropped and $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ refer to the parameter-free forms given by (3.5) and (3.8).

4. The generators of the five-parameter subgroup

Corresponding to the two linear and three quadratic invariants there is a five-parameter group which is a subgroup of the complete group. Before proceeding to obtain the generators of the subgroup, the five invariants are written down in the order and form which corresponds to the usage of Lutzky (1978) and Leach (1979). This will facilitate comparison. (See also the discussion at the end of § 6.) Thus

$$
\begin{aligned}
\phi_1(q, p, t) &= -[C_2]_{12} \\
&= \tfrac{1}{2}[(q^2 + p^2) \sinh 2t - 2qp \cosh 2t]
\end{aligned}
\tag{4.1a}
$$

$$
\begin{aligned}
\phi_2(q, p, t) &= -\tfrac{1}{2}([C_2]_{11} - [C_2]_{22}) \\
&= -\tfrac{1}{2}(q^2 - p^2)
\end{aligned}
\tag{4.1b}
$$

$$
\begin{aligned}
\phi_3(q, p, t) &= -C_{12} \\
&= -(-q \sinh t + p \cosh t)
\end{aligned}
\tag{4.1c}
$$

$$
\begin{aligned}
\phi_4(q, p, t) &= C_{11} \\
&= q \cosh t - p \sinh t
\end{aligned}
\tag{4.1d}
$$

$$
\begin{aligned}
\phi_5(q, p, t) &= \tfrac{1}{2}([C_2]_{11} + [C_2]_{22}) \\
&= -\tfrac{1}{2}[(q^2 + p^2) \cosh 2t + 2qp \sinh 2t].
\end{aligned}
\tag{4.1e}
$$

The generator of a one-parameter group is given by

$$
G(q, t) = \xi(q, t)\, \partial/\partial t + \eta(q, t)\, \partial/\partial q \tag{4.2}
$$

and the corresponding invariant, in the Lagrangian formulation, is

$$
\phi(q, \dot{q}, t) = (\xi \dot{q} - \eta)\, \partial L/\partial \dot{q} - \xi L + f(q, t). \tag{4.3}
$$

In the Hamiltonian formulation, making use of

$$
p = \partial L/\partial \dot{q}, \qquad L = p\dot{q} - H, \tag{4.4}
$$

the invariant is

$$
\phi(q, p, t) = \xi H - \eta p + f(q, t). \tag{4.5}
$$

With H as given by (2.2) and in the order corresponding to the listing of the invariants in (4.1*a*–*e*), ξ, η and $f(q, t)$ are given by

$\xi(q, t)$	$\eta(q, t)$	$f(q, t)$	
$\sinh 2t$	$q \cosh 2t$	$q^2 \sinh 2t$	
1	0	0	
0	$\cosh t$	$q \sinh t$	
0	$\sinh t$	$q \cosh t$	
$\cosh 2t$	$q \sinh 2t$	$q^2 \cosh 2t$	(4.6)

and the generators are

$$G_1 = \sinh 2t\, \partial/\partial t + q \cosh 2t\, \partial/\partial q \tag{4.7a}$$

$$G_2 = \partial/\partial t \tag{4.7b}$$

$$G_3 = \cosh t\, \partial/\partial q \tag{4.7c}$$

$$G_4 = \sinh t\, \partial/\partial q \tag{4.7d}$$

$$G_5 = \cosh 2t\, \partial/\partial t + q \sinh 2t\, \partial/\partial q. \tag{4.7e}$$

It is noted that for the first, second and fifth,

$$\xi = \frac{\partial^2}{\partial q^2}[2F(q, t)], \qquad \eta = \frac{\partial^2}{\partial q\, \partial t}[F(q, t)], \qquad f = \frac{\partial^2}{\partial t^2}[F(q, t)] \tag{4.8}$$

with $F(q, t)$ being $\frac{1}{4}q^2 \sinh 2t$, $\frac{1}{4}q^2$ and $\frac{1}{4}q^2 \cosh 2t$ respectively. For the third and fourth,

$$\xi = \frac{\partial^2}{\partial q^2}[F(q, t)], \qquad \eta = \frac{\partial^2}{\partial q\, \partial t}[F(q, t)], \qquad f = \frac{\partial^2}{\partial t^2}[F(q, t)] \tag{4.9}$$

with $F(q, t)$ being $q \sinh t$ and $q \cosh t$ respectively.

5. The generators of the three-parameter subgroup

The maximum number of one-parameter groups for a second-order differential equation is eight (Bluman and Cole 1974). Of these, five have been obtained using the Hamiltonian formulation and transformation theory. The three remaining are obtained using the method of extended Lie groups by applying the second extension of the generator to the Newtonian equation of motion. This equation of motion is

$$\ddot{q} - q = 0. \tag{5.1}$$

The second extension of the operator

$$G(q, t) = \xi(q, t)\, \partial/\partial t + \eta(q, t)\, \partial/\partial q \tag{5.2}$$

is (Prince and Eliezer 1980)

$$G^{(2)}(q, t) = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \eta^{(1)} \frac{\partial}{\partial \dot{q}} + \eta^{(2)} \frac{\partial}{\partial \ddot{q}} \tag{5.3}$$

where

$$\eta^{(1)} = d\eta/dt - \dot{q}\ d\xi/dt \tag{5.4}$$

$$\eta^{(2)} = d\eta^{(1)}/dt - \ddot{q}\ d\xi/dt \tag{5.5}$$

$$d/dt = \partial/\partial t + \dot{q}\ \partial/\partial q. \tag{5.6}$$

G is the generator of a one-parameter group provided

$$G^{(2)}(\ddot{q} - q) = 0 \tag{5.7}$$

whenever equation (5.1) is satisfied. This gives rise to the set of partial differential equations

$$\xi_{qq} = 0 \tag{5.8a}$$

$$\eta_{aq} - 2\xi_{qt} = 0 \tag{5.8b}$$

$$2\eta_{qt} - \xi_{tt} - 3q\xi_q = 0 \tag{5.8c}$$

$$\eta_{tt} - \eta + q\eta_q - 2q\xi_t = 0. \tag{5.8d}$$

The solution of these equations is not particularly difficult in this case. However, for some other problems it is not a trivial task. One way to avoid having to solve these equations is to compare the known solutions for G_1 to G_5 already obtained with those of a similar problem. In this case the attractive oscillator is suitable for comparison. It is observed that for G_1 to G_5, in the coefficients of the $\partial/\partial t$ terms, cos goes to cosh and sin to sinh while for the $\partial/\partial q$ terms cos goes to cosh and sin to −sinh. For the attractive oscillator (cf Lutzky (1978)),

$$G_6 = q\ \partial/\partial q \tag{5.9a}$$

$$G_7 = q \sin t\ \partial/\partial t + q^2 \cos t\ \partial/\partial q \tag{5.9b}$$

$$G_8 = q \cos t\ \partial/\partial t - q^2 \sin t\ \partial/\partial q. \tag{5.9c}$$

This suggests that for the repulsive oscillator

$$\xi_6 = 0, \qquad \eta_6 = q \tag{5.10a}$$

$$\xi_7 = q \sinh t, \qquad \eta_7 = q^2 \cosh t \tag{5.10b}$$

$$\xi_8 = q \cosh t, \qquad \eta_8 = q^2 \sinh t. \tag{5.10c}$$

Each of these pairs in turn satisfies the equations (5.8*a*–*d*). Therefore it is proposed that for the repulsive oscillator

$$G_6 = q\ \partial/\partial q \tag{5.11a}$$

$$G_7 = q \sinh t\ \partial/\partial t + q^2 \cosh t\ \partial/\partial q \tag{5.11b}$$

$$G_8 = q \cosh t\ \partial/\partial t + q^2 \sinh t\ \partial/\partial q. \tag{5.11c}$$

6. The complete symmetry group

The final test for the generators is whether they have commutation relations appropriate to one of the established groups. One reason for this is to identify the group. Another reason is to check the accuracy of the expressions for the generators, especially those of G_6, G_7 and G_8 in view of the method used to obtain them. The commutation relations are given in table 1. These are the standard relations, as reported in recent literature, for the symmetry group Sl(3, R) which is therefore the complete symmetry group for the repulsive oscillator.

Table 1. The commutation relations. The (i, j)th entry is the bracket $[G_i, G_j]$.

G_i \ G_j	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	$-2G_5$	$-G_3$	G_4	$-2G_2$	0	G_7	$-G_8$
G_2	$2G_5$	0	G_4	G_3	$2G_1$	0	G_8	G_7
G_3	G_3	$-G_4$	0	0	G_4	G_3	$\frac{1}{2}(G_1+3G_6)$	$\frac{1}{2}(G_2+G_5)$
G_4	$-G_4$	$-G_3$	0	0	$-G_3$	G_4	$-\frac{1}{2}(G_2-G_5)$	$\frac{1}{2}(G_1-36_6)$
G_5	$2G_2$	$-2G_1$	G_4	G_3	0	0	G_8	$-G_7$
G_6	0	0	$-G_3$	$-G_4$	0	0	G_7	G_8
G_7	$-G_7$	$-G_8$	$-\frac{1}{2}(G_1+3G_6)$	$\frac{1}{2}(G_2-G_5)$	$-G_8$	$-G_7$	0	0
G_8	G_8	$-G_7$	$-\frac{1}{2}(G_2+G_5)$	$-\frac{1}{2}(G_1-3G_6)$	G_7	$-G_8$	0	0

Those who are familiar with the form of the generators for the attractive oscillator will have observed that the operators G_2 and G_5 appear to have been interchanged for the repulsive oscillator. This is not the case. The operators G_2 and G_5 for the repulsive oscillator are respectively the counterparts of G_2 and G_5 for the attractive oscillator. This is most easily seen from the corresponding quadratic invariants (see Leach 1978b). Under the transformation from attractive to repulsive oscillator, the quadratic invariants transform as

$$(q^2-p^2)\sin 2t+2qp\cos 2t \rightarrow -(q^2+p^2)\sinh 2t+2qp\cosh 2t \qquad (6.1a)$$

$$(q^2-p^2)\cos 2t-2qp\sin 2t \rightarrow q^2-p^2 \qquad (6.1b)$$

$$q^2+p^2 \rightarrow (q^2+p^2)\cosh 2t-2qp\sinh 2t. \qquad (6.1c)$$

For the attractive oscillator G_5 corresponds to the Hamiltonian, whereas for the repulsive oscillator it is G_2 which corresponds to the Hamiltonian. The Hamiltonian (in a time-independent context) is the generator of time translations and so it is appropriate that the generator corresponding to the Hamiltonian in each case is $\partial/\partial t$. In order to preserve the pattern of the commutation relations of the generators for Sl(3, R) in the form adopted by recent writers, the generator of pure time translations may vary from problem to problem. For the attractive oscillator it is G_5 and for the repulsive oscillator discussed here it is G_2. In the case of the alternative form for the Hamiltonian of the repulsive oscillator, namely

$$H = pq, \qquad (6.2)$$

G_1 becomes $\partial/\partial t$.

7. Comment

The complete symmetry group for the one-dimensional attractive oscillator, the free particle and the repulsive oscillator is Sl(3, R) in each instance. For the one-dimensional time-dependent attractive oscillator it has been demonstrated by Leach (1979) that the complete symmetry group is also Sl(3, R). This is also the case for the one-dimensional time-dependent repulsive oscillator for which the generators are given in the Appendix. For an n-dimensional uncoupled time-dependent attractive oscillator system, Prince and Eliezer (1980) have shown that the complete symmetry group is Sl($n+2$, R). From the results summarised above, it may be inferred that the complete symmetry group of an n-dimensional linear system is Sl($n+2$, R) provided that the system is uncoupled, undamped and unforced.

The question now arises as to whether the complete symmetry group of any n-dimensional linear system is also Sl($n+2$, R). It is to be reported elsewhere (Leach (1979) *The Complete Symmetry Group of a One-dimensional Forced Harmonic Oscillator* (unpublished)) that the result holds if forcing is present. The situation with respect to damping is the subject of current investigation. For a coupled system the position is as yet obscure, but it is hoped that some indication will be forthcoming in the near future.

Appendix

The one-dimensional time-dependent repulsive oscillator is described by the Newtonian equation of motion

$$\ddot{q} - \omega^2(t)q = 0 \tag{A1}$$

and has the Hamiltonian

$$H = \tfrac{1}{2}(p^2 - \omega^2(t)q^2). \tag{A2}$$

The Hamiltonian (A2) is related to the corresponding time-dependent Hamiltonian

$$\bar{H} = \tfrac{1}{2}(P^2 - Q^2) \tag{A3}$$

by the linear canonical transform

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 1/\rho, & 0 \\ -\dot{\rho}, & \rho \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{A4}$$

where $\rho(t)$ is any solution of the 'auxiliary equation' (cf Eliezer and Gray 1976)

$$\ddot{\rho} - \omega^2(t)\rho = 1/\rho^3. \tag{A5}$$

The generators of the complete symmetry group Sl(3, R) are

$$G_1 = \sinh 2W\, \partial/\partial W + (-\rho\dot{\rho} \sinh 2W + \cosh 2W)q\, \partial/\partial q \tag{A6a}$$

$$G_2 = \partial/\partial W + \rho\dot{\rho}q\, \partial/\partial q \tag{A6b}$$

$$G_3 = \rho \cosh W\, \partial/\partial q \tag{A6c}$$

$$G_4 = \rho \sinh W\, \partial/\partial q \tag{A6d}$$

$$G_5 = \cosh 2W\, \partial/\partial W + (\rho\dot{\rho} \cosh 2W + \sinh 2W)q\, \partial/\partial q \tag{A6e}$$

$$G_6 = q\, \partial/\partial q \tag{A6f}$$

$$G_7 = \rho^{-1}q \sinh W\, \partial/\partial W + (-\dot{\rho} \sinh W + \rho^{-1} \cosh W)q^2\, \partial/\partial q \tag{A6g}$$

$$G_8 = \rho^{-1} q \cosh W\, \partial/\partial W + (\dot{\rho} \cosh W + \rho^{-1} \sinh W) q^2\, \partial/\partial q \qquad \text{(A6}h\text{)}$$

where

$$W = \int_{t_0}^{t} \rho^{-2}\, dt' \qquad \text{(A7)}$$

is the effective time variable.

Acknowledgment

The writer thanks Professor C J Eliezer and G E Prince for many stimulating discussions on the subject of complete symmetry groups.

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